On geodesics in a stationary spinning string spacetime

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1) Introduction

An axially symmetric spinning source ⟶ topological frame dragging effect
L. Herrera et al., gr-qc/0001075; U. Camci, gr-qc/0507085.

- Some parameters are related to topological defects (the geometry is locally Minkowskian) - only a deficit angle is produced by the mass per unit length of the string.
- cosmic string - thin tube of false vacuum.
  ↓
- a massless spinning cosmic string - a gravitational analog of the Aharonov-Bohm solenoid (P.O. Mazur, hep-th/9611206, 9602044).
- a nonvanishing Sagnac time delay even for a nonrotating

observer, due to the angular momentum of the source.

(M. Ruggiero, gr-qc/0510047; G. Rizzi and M. Ruggiero, gr-qc/0305046).

- the stationary cylindrically symmetric spacetime:

$$ds^2 = -f\,dt^2 + 2k\,dt\,d\varphi + e^{\mu}(dr^2 + dz^2) + l\,d\varphi^2 \quad (1)$$

f, k, μ, l – depend only on r.

$\mu = 0$, (1) is Minkowskian.

and some parameters related to:

- the Newtonian mass per unit length of the line mass.
- an arbitrary constant gravitational potential (M.F.A da Silva et al., gr-qc/9607056).
- the intrinsic angular momentum of the spinning string, which produces a topological frame dragging.
- the vorticity of the source.

$\mu = 0$, $R^{\alpha}{}_{\mu\nu\beta} = 0$, $\Rightarrow \exists$ coordinate transformation that leads from (1) to Minkowski spacetime.

2) The rotating observer.

$(t', r', z', \varphi') \xrightarrow{t' = \bar{t} + b\bar{\varphi}} (2)$:

$$ds^2 = -d\bar{t}^2 - 2b\, d\bar{t}\, d\bar{\varphi} + d\bar{r}^2 + d\bar{z}^2 + (\bar{r}^2 - b^2) d\bar{\varphi}^2 \qquad (2)$$

with $b \sim$ Planck's length.
$b \neq 0$, we have closed timelike curves (CTC).
With $(\bar{t}, \bar{r}, \bar{z}, \bar{\varphi}) \xrightarrow{\bar{\varphi} = -\omega t + \varphi} (t, r, z, \varphi)$,
(2) becomes:

$$ds^2 = -\left[(1-b\omega)^2 - \omega^2 r^2\right] dt^2 - 2\left[b + \omega(r^2 - b^2)\right] dt\, d\varphi + dr^2 + dz^2 + (r^2 - b^2)\, d\varphi^2 \qquad (3)$$

with $\omega = $ const.

$b = 0$, $\rightarrow$ (3) represents a uniformly rotating disk in Minkowski space.
D. Soler, gr-qc/0511041 ; L. D. Landau, E. M. Lifshitz, Teoriia polia (1960).
$r = 1/\omega$ is the light cylinder ($b = 0$).

3) Horizon and surface gravity.

$b \neq 0$, event horizon, from

$$\hat{g}_{00} \equiv g_{00} - \frac{g_{03}^2}{g_{33}} = 0 \qquad (4)$$

whence $r_H = 0$.

$g_{33} = r^2 - b^2 < 0$, the time machine region.

(M. Cvetic et al, hep-th/0504080).

The surface gravity

$$\varkappa^2 = \nabla_\alpha \Lambda \, \nabla^\alpha \Lambda \,, \tag{5}$$

$$\Lambda^2 = -g_{00} - 2\,\Omega_H\, g_{03} - \Omega_H^2\, g_{33} \tag{6}$$

Ω_H – the horizon angular velocity

$$\Omega_H \equiv - \left.\frac{g_{03}}{g_{33}}\right|_{r=0} = \omega - \frac{1}{b} \tag{7}$$

We obtain $\Lambda = -r^2/b^2$, whence $\varkappa^2 = -1/b^2$.

$r_H = 0$, inside the time machine region.

4) Geodesics.

The Lagrangean

$$\mathcal{L} = \frac{1}{2}\, g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \cdot \frac{dx^\beta}{d\lambda} \tag{8}$$

gives for the metric (3)

$$2\mathcal{L} = -\left[(1-b\omega)^2 - \omega^2 r^2\right]\dot t^2 - 2\left[b + \omega(r^2-b^2)\right]\dot t\,\dot\varphi + \dot r^2 + (r^2-b^2)\,\dot\varphi^2 \tag{9}$$

From the Euler-Lagrange eqs.

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} - \frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot x^\alpha}\right) = 0 \,,$$

we obtain the canonical momenta

$$p_t = g_{00}\dot{t} + g_{03}\dot{\varphi} \equiv E$$
$$p_\varphi = -g_{03}\dot{t} + g_{33}\dot{\varphi} \equiv L \qquad (10)$$

Solving for $\dot{t}$ and $\dot{\varphi}$, we have

$$\dot{t} = \bar{E} - \frac{b\bar{L}}{r^2}, \quad \dot{\varphi} = \omega\bar{E} + \frac{(1-b\omega)\bar{L}}{r^2} \qquad (11)$$

with $\bar{E} = E - \omega L$ and $\bar{L} = L + bE$

In addition

$$\varepsilon = \cancel{\boxed{\;}} = g_{00}\dot{t}^2 + 2g_{03}\dot{t}\dot{\varphi} - \dot{r}^2 - g_{33}\dot{\varphi}^2 \qquad (12)$$

where $\varepsilon = \pm 1; 0$.

Putting (11) in (12), one obtains

$$\dot{r}^2 + \frac{\bar{L}^2}{r^2} = \bar{E}^2 - \varepsilon. \qquad (13)$$

From eqs. (11) and (13):

$$\frac{\dot{r}}{\dot{t}} \equiv \frac{dr}{dt} = \pm\left(\bar{\varepsilon}^2 - \frac{K^2}{r^2}\right)^{1/2} \cdot \left(1 - \frac{Kb}{r^2}\right)^{-1} \qquad (14)$$

with $\bar{\varepsilon}^2 = 1 - \frac{\varepsilon}{\bar{E}^2}$ and $K = \frac{\bar{L}}{\bar{E}}$.

$\pm \longrightarrow$ out (in) going geodesics.

An integration gives

$$\sqrt{r^2-a^2} - b\bar{\varepsilon}\arctan\frac{\sqrt{r^2-a^2}}{a} = \pm\bar{\varepsilon}t + \text{const.} \quad (15)$$

with $a = K/\bar{\varepsilon}$. ~~and $r(0)=a$~~

a) Outgoing radial timelike geodesics (ORTG)
$\varepsilon=1$, $L=0$, $dr/dt>0$. Therefore
$\bar{\varepsilon}=p/E$, $K=b$, $a=bE/p$, $r\geq a>b$.
From (15) we obtain

$$\frac{dr}{dt} = \frac{b}{a}\cdot\frac{r\sqrt{r^2-a^2}}{r^2-b^2} \quad (16)$$

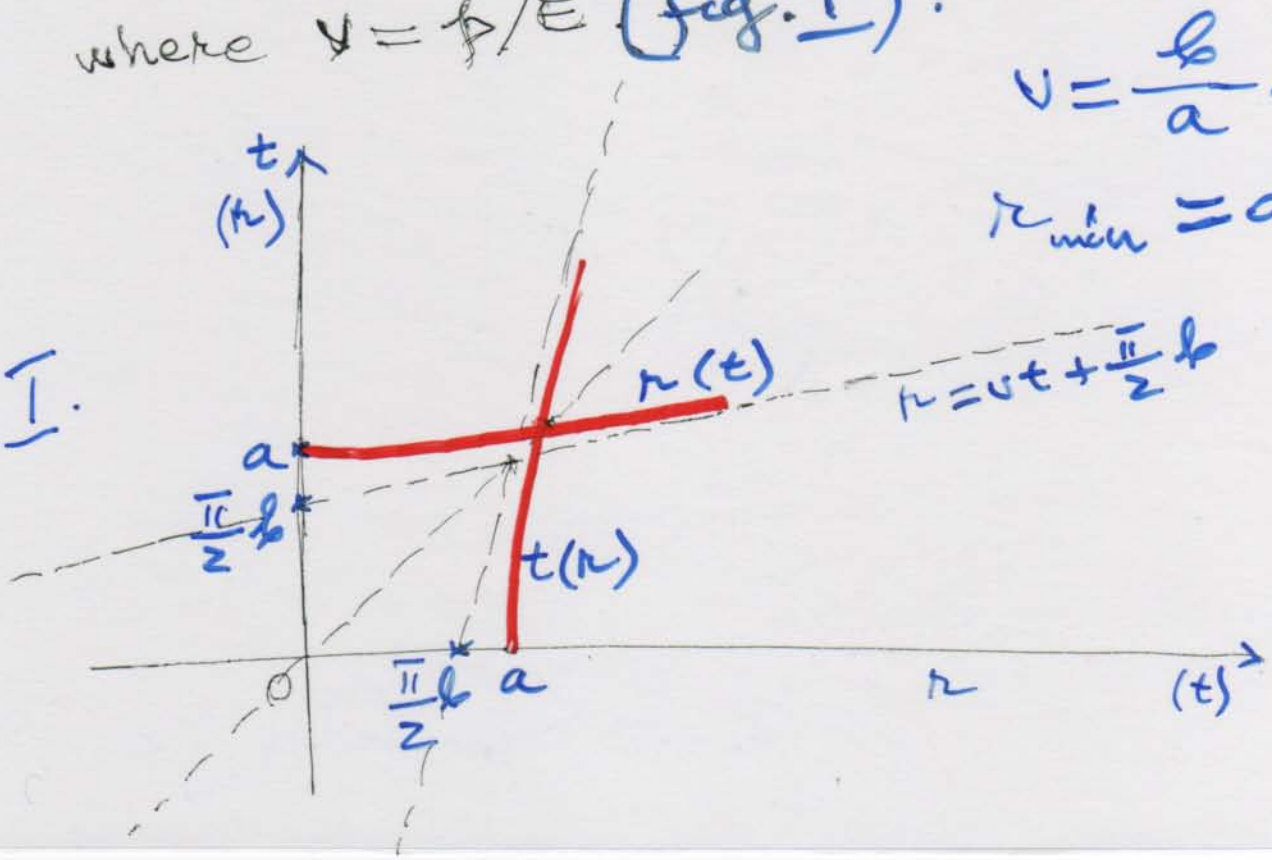
$r(t)$ tends asymptotically to the straight line

$$r = vt + \frac{\pi}{2}b \quad (17)$$

where $v = p/E$ (fig. I).

$v = \frac{b}{a}$.

$r_{min} = a$

I. 

b) IRTG.

$\varepsilon = 1$, $L = 0$, $dr/dt < 0$.

Take $r = \rho > a$ at $t = 0$. From (14) and (15) we get

$$t = \frac{a}{b}\left(\sqrt{\rho^2 - a^2} - \sqrt{r^2 - a^2}\right) + b\left(\arctan\frac{\sqrt{r^2-a^2}}{a} - \arctan\frac{\sqrt{\rho^2-a^2}}{a}\right) \qquad (18)$$

For any $\rho > a$, we have $a \leq r \leq \rho$. (fig. II)

Example:

We have $a \cong b/v$. With $v \sim 300$ cm/s, one obtains $a \sim 10^8 b \approx 10^{-25}$ cm. Hence, "a" has microscopic values for reasonable v and, therefore, the trajectory overlaps its asymptote after a very short time.

$r_{min} = a$

$$\left(\frac{dr}{dt}\right)_{r=a} = 0.$$

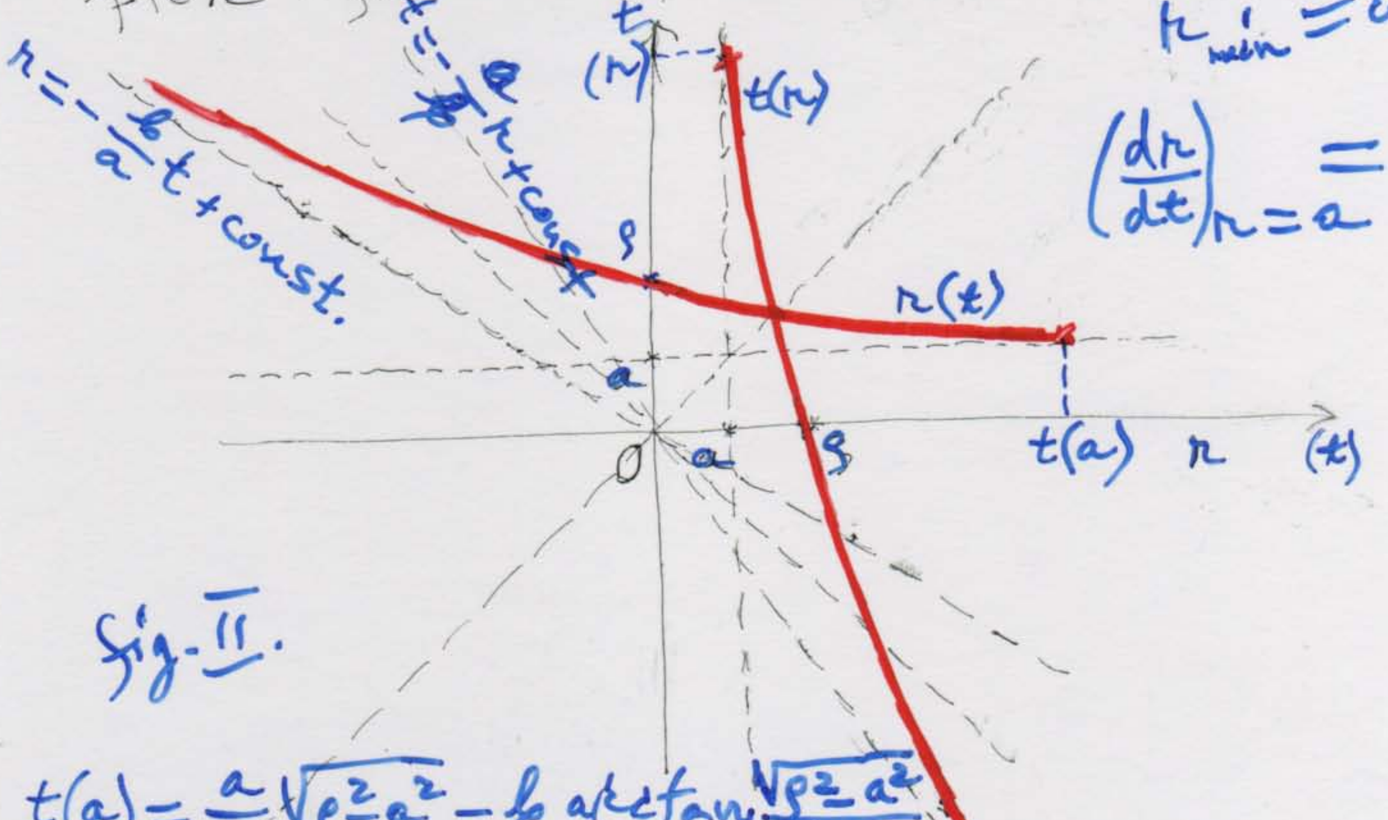


fig. II.

$$t(a) = \frac{a}{b}\sqrt{\rho^2 - a^2} - b\arctan\frac{\sqrt{\rho^2-a^2}}{a}$$

c) ONRG.

We have now $\varepsilon = 0$, $L = 0$, $dr/dt > 0$.
Take $r = d > b$ at $t = 0$.
The equation of motion is

$$t = \sqrt{r^2 - b^2} - \sqrt{d^2 - b^2} - b\left(\arctan\frac{\sqrt{r^2 - b^2}}{b} - \right.$$
$$\left. - \arctan\frac{\sqrt{d^2 - b^2}}{b}\right) \qquad (19)$$

with
$$\frac{dr}{dt} = \frac{r}{\sqrt{r^2 - b^2}} \qquad (20)$$

Hence, $(dr/dt)_{t=0} = d/\sqrt{d^2 - b^2} > 1$

and then $r(t)$ approaches

$$r = t + \sqrt{d^2 - b^2} + \frac{b\pi}{2} - \arctan\frac{\sqrt{d^2 - b^2}}{b}$$

asymptotically (fig. III).

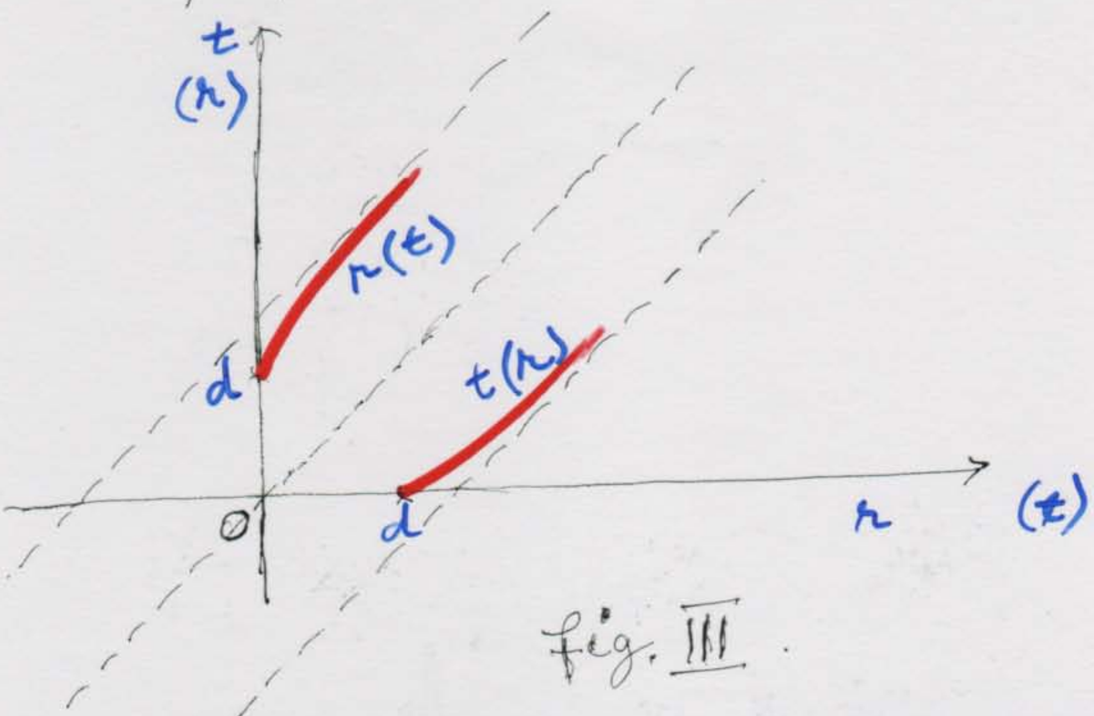


fig. III.

The test particle acquires a tachyonic motion only for very short time, provided $d \gg b$.

d) iNRG.

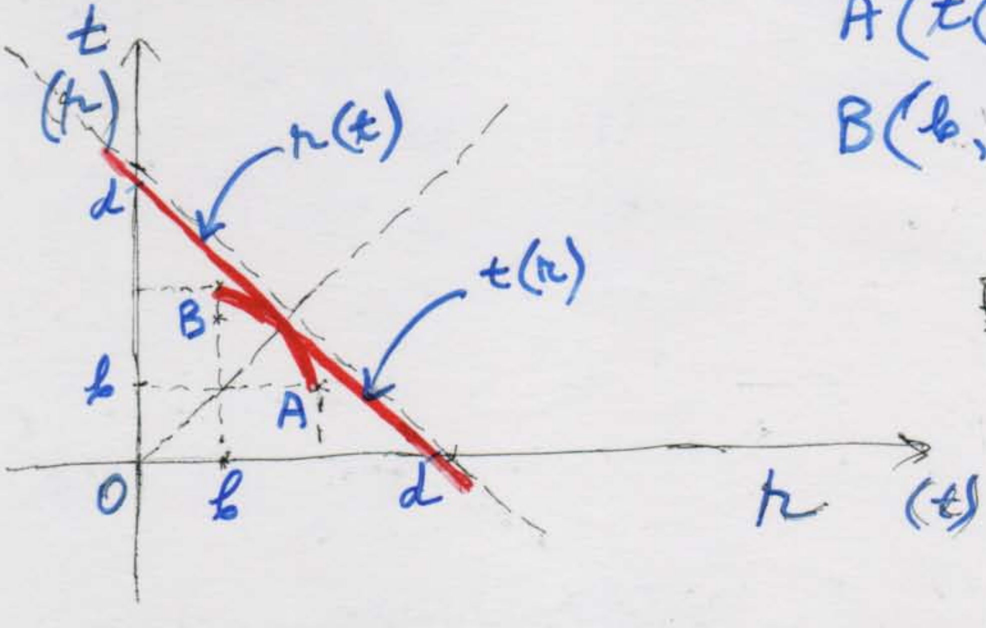
We have $\varepsilon = 0$, $L = 0$, $dr/dt < 0$. Using the same boundary conditions, the curves $r(t)$ are obtained from

$$t = -\sqrt{r^2 - b^2} + \sqrt{d^2 - b^2} + b\left(\arctan\frac{\sqrt{r^2-b^2}}{b} - \right.$$
$$\left. - \arctan\frac{\sqrt{d^2-b^2}}{b}\right) \qquad (21)$$

with $b \le r \le d$, when

$$\sqrt{d^2 - b^2} - b\arctan\frac{\sqrt{d^2-b^2}}{b} \ge t \ge 0.$$

The null particle approaches $r = b$ (the time machine region) with an infinite velocity (fig. IV).



$A(t(b), b)$
$B(b, t(b))$.

Fig. IV.

When $d \gg b$, the particle reaches $r = b$ after $t \simeq d - b\arctan(d/b) \simeq d - \pi b/2$.

Practically, the trajectory deviates from the "classical" ($b = 0$) case only near the time machine limit.

The angular trajectory $\phi(t)$ could be obtained from eqs. (11) which, for $L=0$, become:

$$\dot{\varphi} = \omega E + \frac{b(1-b\omega)E}{r^2} \; ; \; \dot{t} = E\left(1-\frac{b^2}{r^2}\right),$$

whence

$$\frac{d\varphi}{dt} = \omega + \frac{b}{r^2-b^2} \; ; \; r = r(t) \tag{22}$$

5) Conclusions.

- b (related to the angular momentum of the string) $\neq 0$, $\Longrightarrow r_H = 0$ and
- $\Omega_H = \omega - 1/b$.
- $\varkappa$ (surface gravity) is imaginary, with $|\varkappa| = 1/b$.
- Parameters: E, L, b and ω.
- When $L=0$, $r(t)$ does not depend on ω.
- bE — plays the role of an intrinsic spin.

- for the **Sagnac effect** we have, with respect to the asymptotically rest frame (MTW, Gravitation, 1973):

$$\Omega_+ \equiv \left(\frac{d\varphi}{dt}\right)_+ = \omega + \frac{1}{r_0 - b} \tag{23a}$$

$$\Omega_- \equiv \left(\frac{d\varphi}{dt}\right)_- = \omega - \frac{1}{r_0 + b} \tag{23b}$$

with the time delay

$$\Delta t = 4\pi b \left[(1-b\omega)^2 - \omega^2/r_0^2\right]^{-1} \tag{24}$$

$\omega = 0, \Longrightarrow \Delta t = 4\pi b \neq 0.$